The general solution of given differential equation is given by

$$F_1(x,y,c_1)\,F_2(x,y,c_2)\ \ldots\ldots\ F_n(x,y,c_n)=0$$

Since the given differential equation is of first order ; therefore , the g... can not have more than one arbitrary constant , so we take

$$c_1=c_2=c_3\ldots\ldots\ldots\ldots c_n=c\ \text{(say)}$$

Hence the general solution of the given equation can be put as

$$F_1(x,y,c)\,F_2(x,y,c)\ \ldots\ldots\ F_n(x,y,c)=0$$

Working Rule :

1. Reduce the given differential equation in linear factors of p
2. Solve individually these linear factors of p
3. Multiply all the solutions obtained in step 2 and equate it to zero, ... general solution of the given differential equation.

Illustrative Examples

Example 1 : Solve $p^2 - 7p + 12 = 0$

Solution : The given differential equation is

$$p^2-7p+12=0.$$

or $\quad (p-3)(p-4)=0$

$\Rightarrow\quad p=3 \quad$ or $\quad p=4$

i.e., $\quad \dfrac{dy}{dx}=3 \quad$ or $\quad \dfrac{dy}{dx}=4$

$\Rightarrow\quad dy=3\,dx \qquad dy=4\,dx$

Integrating, $\quad y=3x+c \qquad$ Integrating , $\quad y=4x+c$

or $\quad y-3x-c=0 \qquad$ or $\quad y-4x-c=0$

Therefore the general solution of given equation is

$$(y-3x-c)(y-4x-c)=0.$$

Example 2 : Solve $p^2 + 2py\cot x = y^2$

Solution : The given differential equation is $p^2+2py\cot x=y^2$

Adding $y^2\cot^2 x$ both sides, we get

$$p^2+2py\cot x+y^2\cot^2 x=y^2+y^2\cot^2 x$$

$$\Rightarrow\quad (p+y\cot x)^2=y^2(1+\cot^2 x)=y^2\operatorname{cosec}^2 x$$

$$\Rightarrow\quad p+y\cot x=\pm y\operatorname{cosec} x$$

Thus, the component equations are

$$p=y(-\cot x+\operatorname{cosec} x)$$

and

$$p=y(-\cot x-\operatorname{cosec} x)$$

Now from equation (2), $p=y(\operatorname{cosec} x-\cot x)$

or
$$\frac{dy}{dx}=y\left(\frac{1-\cos x}{\sin x}\right)=y\,\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\,\cos\frac{x}{2}}$$

or
$$\frac{dy}{dx}=y\tan\frac{x}{2}$$

$$\Rightarrow \qquad \frac{dy}{y} = \tan\frac{x}{2}dx$$

Integrating both sides, we get

$$\log y = 2\log\sec\frac{x}{2} + \log c'$$

i.e.,
$$\log y = \log\left[c' \sec^2\frac{x}{2}\right]$$

or
$$y = c'\sec^2\frac{x}{2}$$

or
$$y\cos^2\frac{x}{2} = c'$$

or $\quad y(1+\cos x) = c \qquad$ where $c = 2c'$ ......(4)

Similarly from equation (3), we have

$$p = -y\left(\frac{1+\cos x}{\sin x}\right)$$

$$\Rightarrow \qquad \frac{dy}{dx} = -y\cot\frac{x}{2} \qquad \Rightarrow \qquad \frac{dy}{y} = -\cot\frac{x}{2}dx$$

Integrating, we get $\log y = -2\log\sin\frac{x}{2} + \log c'$

$$\log y = \log\frac{c'}{\sin^2\frac{x}{2}} \qquad \text{or} \qquad y\sin^2\frac{x}{2} = c'$$

$y(1-\cos x) = c \qquad$ where $\quad c = 2c'$ ......(5)

From equation (4) and (5), general solution is

$$[y(1+\cos x) - c]\,[y(1-\cos x) - c] = 0.$$

Example 3 : Solve $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 - x^2 = 0$.

[K.U. 2013, 15; M.D.U. 2005, 07, 14]

Solution : The given differential equation is

$$x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 - x^2 = 0 \qquad \text{......(1)}$$

Putting $\frac{dy}{dx} = p$, we get $\quad x^2p^2 - 2xyp + 2y^2 - x^2 = 0$

$$\Rightarrow \qquad p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$$

$$= \frac{xy \pm \sqrt{x^2y^2 - 2x^2y^2 + x^4}}{x^2} = \frac{xy \pm x\sqrt{x^2 - y^2}}{x^2}$$

$$\Rightarrow \qquad p = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x} \pm \frac{\sqrt{x^2 - y^2}}{x} \qquad \text{......(2)}$$

Since it is homogeneous equation, so let

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then equation (2) becomes,

$$v + x\frac{dv}{dx} = v \pm \sqrt{1-v^2}$$

or

$$x\frac{dv}{dx} = \pm\sqrt{1-v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1-v^2}} = \pm\frac{dx}{x}$$

Integrating, we get $\sin^{-1}v = \pm \log x + c$

$$\Rightarrow \quad \sin^{-1}\frac{y}{x} = \pm\log x + c \qquad [\because y = vx]$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{y}{x}\right) = \log x + c \quad \text{and} \quad \sin^{-1}\left(\frac{y}{x}\right) = -\log x + c$$

$\therefore$ The general solution of equation (1) is

$$\left(\sin^{-1}\frac{y}{x} - \log x - c\right)\left(\sin^{-1}\frac{y}{x} + \log x - c\right) = 0$$

Example 4 : Solve $x^2p^3 + yp^2(1 + x^2y) + py^3 = 0$

Solution : The given differential equation is

$$x^2p^3 + yp^2(1 + x^2y) + py^3 = 0$$

It may be written as

$$p[x^2p^2 + yp(1 + x^2y) + y^3] = 0$$

$\Rightarrow$ Either $p = 0$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

Integrating, we get

$$y = c$$

or

$$x^2p^2 + y(1 + x^2y)p + y^3 = 0$$

$$\Rightarrow \quad p = \frac{-y(1+x^2y) \pm \sqrt{y^2(1+x^2y)^2 - 4x^2y^3}}{2x^2}$$

$$= \frac{-y(1+x^2y) \pm y\sqrt{(1+x^4y^2+2x^2y) - 4x^2y}}{2x^2}$$

$$= \frac{-y(1+x^2y) \pm y(1-x^2y)}{2x^2}$$

$$\Rightarrow \quad p = \frac{-y}{x^2}, \quad p = -y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x^2}, \quad \frac{dy}{dx} = -y^2$$

$$\frac{dy}{y} = -\frac{dx}{x^2}, \quad \frac{dy}{y^2} = -dx$$

Integrating, we get

$$\log y = \frac{1}{x} + c \quad , \quad \frac{-1}{y} = -x + c$$

$$-1 = -xy + cy$$

Hence general solution is $(y - c)\left(\log y - \frac{1}{x} - c\right)(xy - 1 - cy) = 0.$

Therefore general solution is given by

$$(y-c)(y-x-c)(2y-x^2-2c)(\log y-2x-c)=0$$

Example 7 : Solve the differential equation

$$\left(1-y^2+\frac{y^4}{x^2}\right)p^2-\frac{2y}{x}p+\frac{y^2}{x^2}=0$$

Solution : The given equation can be written as

$$p^2-\frac{2y}{x}p+\frac{y^2}{x^2}=p^2y^2-\frac{p^2y^4}{x^2}$$

$$\Rightarrow \quad \left(p-\frac{y}{x}\right)^2=p^2y^2-\frac{p^2y^4}{x^2}$$

$$\Rightarrow \quad \left(p-\frac{y}{x}\right)^2=p^2y^2\left(1-\frac{y^2}{x^2}\right)$$

$$\Rightarrow \quad px-y=\pm\, py(x^2-y^2)^{1/2}$$

$$\Rightarrow \quad \left[x\mp y(x^2-y^2)^{1/2}\right]\frac{dy}{dx}=y$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{x\mp y(x^2-y^2)^{1/2}}{y}$$

Put $x=vy$, so that $\dfrac{dx}{dy}=v+y\dfrac{dv}{dy}$

Then (1) becomes $\quad v+y\dfrac{dv}{dy}=\dfrac{vy\mp y\sqrt{v^2y^2-y^2}}{y}=v\mp y\sqrt{v^2-1}$

$$\Rightarrow \quad \frac{dv}{dy}=\mp\sqrt{v^2-1}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{v^2-1}}=\mp\, dy$$

Integrating, we get $\quad \log\left(v+\sqrt{v^2-1}\right)=\mp y+c$

$$\Rightarrow \quad \log\left(\frac{x}{y}+\sqrt{\frac{x^2}{y^2}-1}\right)=\mp y+c$$

$$\Rightarrow \quad \log\left(\frac{x+\sqrt{x^2-y^2}}{y}\right)=\mp y+c$$

which is the required solution.

Exercise – 2.1

Solve the following differential equations :

1. $p^2-5p+6=0$ **[M.D.U. 2015]**

2. $x^2\left(\dfrac{dy}{dx}\right)^2+xy\dfrac{dy}{dx}-6y^2=0$

2. EQUATIONS SOLVABLE FOR x :

A differential equation solvable for x can be put in the form

$$x = f(y, p) \qquad \text{......(1)}$$

Differentiating w. r. t. y, we get

$$\frac{dx}{dy} = g\left(y, p, \frac{dp}{dy}\right)$$

$$\Rightarrow \qquad \frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right) \qquad \text{......(2)}$$

This equation involves two variables y and p. Let solution of this equation be

$$\phi(y, p, c) = 0 \qquad \text{......(3)}$$

Elimination of p from (1) and (3) gives the required solution. If elimination of p is not possible, then expressions of x and y in terms of p together form the solution of the given equation.

Working Rule :

1. Express x explicitly in terms of p and y.

2. Differentiate the equation w.r.t. y and replace $\frac{dx}{dy}$ by $\frac{1}{p}$. The equation obtained involves p and y.

3. Solve the equation obtained in step 2 and eliminate p from this solution and the given equation.

Illustrative Examples

Example 1 : Solve $y - 2px = y^2p^3$ **[M.D.U. 2010]**

Solution : The given differential equation is $y = 2px + y^2p^3$ ......(1)

$$\Rightarrow \qquad x = \frac{y}{2p} - \frac{y^2p^2}{2}$$

Differentiating w.r.t. y, we have

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - py^2\frac{dp}{dy}$$

or $$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - py^2\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 = -\left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 = -\frac{y}{p}\left(\frac{1}{2p} + p^2y\right)\frac{dp}{dy}$$

or $$\left(\frac{1}{2p} + yp^2\right) + \frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy} = 0$$

or $$\left(\frac{1}{2p} + yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

or $$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or $$\frac{dy}{y}+\frac{dp}{p}=0$$

Integrating, we get $$\log y+\log p=\log c$$

or $$\log yp=\log c$$

i.e., $$yp=c \quad \Rightarrow \quad p=\frac{c}{y}$$

Thus from equation (1), we have

$$y=\frac{2c}{y}x+y^2\frac{c^3}{y^3}$$

or $$y^2=2cx+c^3$$

which is the general solution of given equation.

Example 2 : Solve $x = 4p + 4p^3$

Solution : The given differential equation is $x = 4p + 4p^3$ ......(1)

Differentiating equation (1) w.r.t. y, we have

$$\frac{dx}{dy}=4\frac{dp}{dy}+12p^2\frac{dp}{dy}$$

$$\frac{1}{p}=(4+12p^2)\frac{dp}{dy}$$

or $$dy=(4p+12p^3)dp$$

Integrating, $$y=2p^2+3p^4+c \quad ......(2)$$

Equation (1) and equation (2) taken together represents the required solution.

Example 3 : Solve $p=\tan\left(x-\frac{p}{1+p^2}\right)$ **[K.U. 2016]**

Solution : The given differential equation is

$$p=\tan\left(x-\frac{p}{1+p^2}\right) \quad ......(1)$$

$$\Rightarrow \quad \tan^{-1}p=x-\frac{p}{1+p^2}$$

$$\Rightarrow \quad x=\tan^{-1}p+\frac{p}{1+p^2} \quad ......(2)$$

Differentiating equation (2) w.r.t. y, we have

$$\frac{dx}{dy}=\frac{1}{1+p^2}\frac{dp}{dy}+\frac{(1+p^2).1-p.2p}{(1+p^2)^2}\frac{dp}{dy}.$$

or $$\frac{1}{p}=\left(\frac{1}{1+p^2}+\frac{1-p^2}{(1+p^2)^2}\right)\frac{dp}{dy}$$

or $$dy=\frac{2p}{(1+p^2)^2}dp$$

Integrating, $$y=\frac{(1+p^2)^{-1}}{-1}+c$$

or $$y = c - \frac{1}{1+p^2}$$

Therefore general solution is given by

$$x = \tan^{-1} p + \frac{p}{1+p^2}, \quad y = c - \frac{1}{1+p^2}$$

Exercise – 2.2

Solve the following differential equations :

1. $y = 2px + p^2 y$ **2.** $x = y + p^2$ **3.** $x = y + a\log p$ **4.** $y = 3px + 6p^2y^2$

5. $y^2 \log y = xyp + p^2$ **6.** $p^3 - 4xyp + 8y^2 = 0$ **[K.U. 2011]** **7.** $yp^2 - 2xp + y = 0$

Answers

1. $y^2 = 2cx + c^2$ **2.** $x = c - [2p + 2\log(p-1)]$, $y = c - [p^2 + 2p + 2\log(p-1)]$.

3. $y = c - a\log(1-p)$, $x = c - a\log(1-p) + a\log p$ **4.** $y^3 = 3cx + 6c^2$

5. $\log y = c^2 + cx$ **6.** $64y = c(4x-c)^2$ **7.** $y^2 = 2cx - c^2$

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3. EQUATIONS SOLVABLE FOR y :

A differential equation solvable for y can be put in the form

$$y = f(x, p) \qquad \text{.......(1)}$$

Differentiating w.r.t. y, we get

$$\frac{dy}{dx} = g\left(x, p, \frac{dp}{dx}\right)$$

$$\Rightarrow \qquad p = g\left(x, p, \frac{dp}{dx}\right) \qquad \text{.......(2)}$$

This equation involves two variables x and p. Let solution of this equation be

$$\phi(x, p, c) = 0 \qquad \text{.......(3)}$$

Elimination of p from (1) and (3) gives the required solution. If elimination of p is not possible, then expressions of x and y in terms of p together form the solution of the given equation.

Working Rule :

1. Express y explicitly in terms of p and x.
2. Differentiate the equation w.r.t. x and replace $\frac{dy}{dx}$ by p. The obtained equation involves p and x.
3. Solve the equation obtained in step 2 and eliminate p from this solution and the given equation.

Illustrative Examples

Example 1 : Solve $y = 3x + \log p$ **[M.D.U. 2006]**

Solution : The given differential equation is

$$y = 3x + \log p \qquad \text{......(1)}$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$p - 3 = \frac{1}{p}\frac{dp}{dx}$$

or $$dx = \frac{dp}{p(p-3)}$$

Integrating both sides, we have

$$x = \frac{1}{3}\int\left(-\frac{1}{p} + \frac{1}{p-3}\right)dp + c' \qquad \text{[Making partial fractions]}$$

or $$x = \frac{1}{3}\left[-\log p + \log(p-3)\right] + c'$$

or $$3x = -\log\frac{p}{p-3} + 3c'$$

or $$3x = -\log\frac{p}{p-3} + \log c'' \qquad [3c' = \log c'']$$

or $$\log e^{3x} = \log c''\left(\frac{p-3}{p}\right) \qquad [\because x = \log e^{x}]$$

or $$e^{3x} = c''\frac{p-3}{p}$$

or $$e^{3x} = c''\left(1 - \frac{3}{p}\right)$$

or $$1 - ce^{3x} = \frac{3}{p} \quad \text{or} \quad p = \frac{3}{1 - ce^{3x}} \qquad \left[\because \frac{1}{c''} = c\right]$$

Then from equation (1), we have $y = 3x + \log\frac{3}{1-ce^{3x}}$ which is the required solution.

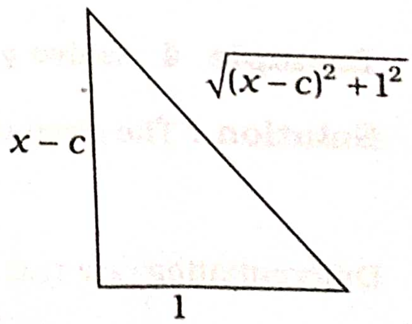
Example 2 : Solve $y = p \tan p + \log \cos p$ **[M.D.U. 2009]**

Solution : The given differential equation is

$$y = p\tan p + \log\cos p \qquad \text{......(1)}$$

$\therefore$ Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = p\sec^2 p\frac{dp}{dx} + \tan p\frac{dp}{dx} - \tan p\frac{dp}{dx}$$



or $$p = p\sec^2 p\frac{dp}{dx}$$

or $$dx = \sec^2 p\, dp$$

Integrating, $$x = \tan p + c$$

or $$p = \tan^{-1}(x - c)$$

Then from equation (1), we have

$$y = (x-c)\tan^{-1}(x-c) - \frac{1}{2}\log\left((x-c)^2+1\right).$$

Example 3 : Solve $y = 2px - p^2$ **[M.D.U. 2013]**

Solution : The given differential equation is $y = 2px - p^2$ ......(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

or $$p = 2p + (2x-2p)\frac{dp}{dx} \quad \text{or} \quad \frac{dx}{dp} = 2 - \frac{2}{p}x$$

or $$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad \text{......(2)}$$

Now equation (2) is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$ where P & Q are functions of x. So $I.F. = e^{\int P dx}$

Here $$I.F. = e^{\int \frac{2}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2$$

Hence solution of (2) is given by

$$x(I.F) = \int 2.(I.F.)dp + c$$

or $$xp^2 = 2\int p^2 dp + c$$

or $$xp^2 = \frac{2p^3}{3} + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2}$$

Then from equation (1), we have

$$y = \frac{4}{3}p^2 + 2cp^{-1} - p^2 = \frac{1}{3}p^2 + 2cp^{-1}$$

Thus the general solution is given by

$$x = \frac{2}{3}p + cp^{-2} \quad \text{and} \quad y = \frac{1}{3}p^2 + 2cp^{-1}$$

Example 4 : Solve $y = xp^2 - \frac{1}{p}$.

Solution : The given differential equation is

$$y = xp^2 - \frac{1}{p} \qquad \text{......(1)}$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = p^2 + 2xp\frac{dp}{dx} + \frac{1}{p^2}\frac{dp}{dx}$$

i.e.,
$$p = p^2 + \left(2xp + \frac{1}{p^2}\right)\frac{dp}{dx}$$
$$p - p^2 = \frac{2xp^3 + 1}{p^2}\frac{dp}{dx}$$
$$p^3 - p^4 = (2xp^3 + 1)\frac{dp}{dx}$$
or
$$\frac{dx}{dp} = \frac{2xp^3}{p^3 - p^4} + \frac{1}{p^3 - p^4}$$
$$\frac{dx}{dp} + \frac{2}{p-1}x = \frac{1}{p^3(1-p)} \qquad \ldots\ldots(2)$$

which is linear differential equation of the type $\frac{dy}{dx} + Py = Q$, so *I.F.* is given by $e^{\int P dx}$

Here
$$I.F. = e^{\int \frac{2}{p-1}dp} = e^{2\log(p-1)} = e^{\log(p-1)^2} = (p-1)^2.$$

Then solution is
$$x(I.F.) = \int \frac{(I.F.).1}{p^3(1-p)}dp$$
$$\Rightarrow \quad (p-1)^2 x = -\int \frac{p-1}{p^3}dp = \int\left(-\frac{1}{p^2} + \frac{1}{p^3}\right)dp + c$$
$$(p-1)^2 x = \frac{1}{p} - \frac{1}{2p^2} + c$$
$$x(p-1)^2 = \frac{2p-1}{2p^2} + c \quad \text{or} \quad x = \frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2}$$

Putting this value in given equation, we have
$$y = p^2\left(\frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2}\right) - \frac{1}{p}.$$
or
$$y = \frac{2p-1}{2(p-1)^2} + \frac{cp^2}{(p-1)^2} - \frac{1}{p}$$

$\therefore$ The general solution is given by
$$x = \frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2}$$
and
$$y = \frac{2p-1}{2(p-1)^2} + \frac{cp^2}{(p-1)^2} - \frac{1}{p}$$

Exercise – 2.3

Solve the following differential equations :

1. $y + px = p^2x^4$ **[K.U. 2012, 11]**
2. $y = 2px + p^4x^2$
3. $p^3 + ap^2 = a(y + ax)$ **[K.U. 2008]**
4. $y = p\sin p + \cos p$
5. $p^3 + p = e^y$ **[K.U. 2015; M.D.U. 2010 (2nd Sem.)]**
6. $y = 2p + \sqrt{1+p^2}$

7. $y = 2px + \psi(xp^2)$ 8. $16x^2 + 2p^2y - p^3x = 0$ 9. $y = p - \log(p^2 - 1)$

10. $p^4x^2 + 2xp - y = 0$ 11. $y = x(1+p) + p^2$ [M.D.U. 2011]

12. $y = -px + x^4p^2$

13. $y = 1 + xp^3$ 14. $y = 2xp + p^2$ 15. $x^2 + p^2x = yp$

16. $4y = x^2 + p^2$ 17. $y = x + a\tan^{-1}p$ 18. $p^2 - py + x = 0$

19. $xp^2 - 2yp - ax = 0$ [K.U. 2016] 20. $y = 2px - xp^2$

21. $x - yp = ap^2$ [M.D.U. 2007]

Answers

1. $y = -\dfrac{c}{x} + c^2$ 2. $y = 2\sqrt{cx} + c^2$

3. $p^3 + \alpha p^2 = a(y + \alpha x)$, $ax + c = \dfrac{3}{2}p^2 - \alpha p + \alpha^2\log(\alpha + p)$

4. $x = c + \sin p$ with given relation.

5. $x = -\dfrac{1}{p} + 2\tan^{-1}p + c,\ y = \log p + \log(1 + p^2)$

6. $x = 2\log p + \log\left(p + \sqrt{1+p^2}\right) + c,\ y = 2p + \sqrt{1+p^2}$

7. $y = 2\sqrt{cx} + \psi(c)$ 8. $2c^2y + 16 - x^2c^3 = 0$

9. $x = \log\dfrac{p(p+1)}{p-1} + c,\ y = p - \log(p^2 - 1)$ 10. $y = c^2 + 2\sqrt{cx}$

11. $x = 2(1-p) + ce^{-p}$, $y = (1+p)\left[2(1-p) + ce^{-p}\right] + p^2$

12. $y = -\dfrac{c}{x} + c^2$ 13. $x^{2/3} = (y-1)^{2/3} + c$

14. $xp^2 = -\dfrac{2}{3}p^3 + c, y = 2xy + p^2$ 15. $x = c\sqrt{p} - \dfrac{1}{3}p^2$, $x^2 + p^2x = yp$

16. $\log(p - x) - \dfrac{x}{p-x} = c$ with given relation

17. $x + c = \dfrac{a}{2}\left[\log(1+p^2)^{-1/2}(p-1) - \tan^{-1}p\right]$, $y = x + a\tan^{-1}p$

3.3 EQUATIONS SOLVABLE FOR y

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus, the equations of this type can be put as $y = f(x, p)$ ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and x.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c),\ y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y + px = x^4p^2$.

Sol. Given equation is $y = -px + x^4p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or
$$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

or
$$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Discarding the factor $(1 - 2px^3)$, which does not involve $\frac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating, $\log p + 2\log x = \log c$

or
$$\log px^2 = \log c$$

or
$$px^2 = c \Rightarrow p = \frac{c}{x^2}.$$

Putting this value of p in (1), we have $y = -\frac{c}{x} + c^2$.

Example 2. *Solve* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

or
$$p + (2x - 2p)\frac{dp}{dx} = 0$$

or
$$p\frac{dx}{dp} + 2x - 2p = 0$$

or
$$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad ...(2)$$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p} dp} = e^{2 \log p} = p^2$$

$\therefore$ The solution of (2) is $x\,\text{I.F.} = \int 2\,\text{I.F.}\,dp + c$

or
$$xp^2 = \int 2p^2 dp + c$$

or
$$xp^2 = \frac{2}{3}p^3 + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2} \qquad ...(3)$$

Putting this value of x in (1), we have $\quad y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$

or
$$y = \frac{1}{3}p^2 + 2cp^{-1}. \qquad ...(4)$$

Equations (3) and (4) together constitute the general solution of (1).

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $xp^2 - 2yp + ax = 0$
2. $y - 2px = \tan^{-1}(xp^2)$
3. $16x^2 + 2p^2y - p^3x = 0$
4. $y = x + 2\tan^{-1} p$
5. $y = 3x + \log p$
6. $x - yp = ap^2$
7. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0.$
8. $y = 2px - xp^2$

Answers

1. $2y = cx^2 + \frac{a}{c}$
2. $y = 2\sqrt{cx} + \tan^{-1} c$
3. $16 + 2c^2y - c^3x^2 = 0$
4. $x = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p + c,\ y = \log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1} p + c$
5. $y = 3x + \log\frac{3}{1 - ce^{3x}}$
6. $x = \frac{p}{\sqrt{1-p^2}}(c + a\sin^{-1} p),\ y = \frac{1}{\sqrt{1-p^2}}(c + a\sin^{-1} p) - ap.$
7. $y = c^2 + 2\sqrt{cx}.$
8. $y = 2\sqrt{cx} - c.$

3.4 EQUATIONS SOLVABLE FOR x

If the equation is solvable for x, we can express x explicitly in terms of y and p. Thus, the equations of this type can be put as $x = f(y, p)$ ...(1)

Differentiating (1) w.r.t. y, we get $\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and y.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1 (p, c), \; y = \phi_2 (p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y = 2px + y^2p^3$.

Sol. Solving for x, we have $x = \dfrac{1}{2}\left(\dfrac{y}{p} - y^2p^2\right)$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - 2yp^2 - 2y^2p\frac{dp}{dy}\right)$$

or
$$2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$$

or
$$p + 2yp^4 + y\frac{dp}{dy} + 2y^2p^3\frac{dp}{dy} = 0 \quad \text{or} \quad p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0$$

or
$$\left(p + y\frac{dp}{dy}\right)(1 + 2yp^3) = 0$$

Discarding the factor $(1 + 2yp^3)$ which does not involve $\dfrac{dp}{dy}$, we have

$$p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, $\quad \log y + \log p = \log c \quad$ or $\quad py = c \quad$ or $\quad p = \dfrac{c}{y}$

Putting this value of p in the given equation, we have

$$y = \frac{2cx}{y} + \frac{c^3}{y} \quad \text{or} \quad y^2 = 2cx + c^3$$

which is the required solution.

Example 2. *Solve* $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x = \tan^{-1} p + \dfrac{p}{1+p^2}$ ...(1)

Differentiating both sides w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

or $$\frac{1}{p} = \frac{2(1+p^2) - 2p^2}{(1+p^2)^2}\frac{dp}{dy} \quad \text{or} \quad dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating, $$y = c - \frac{1}{1+p^2} \qquad ...(2)$$

Equations (1) and (2) together constitute the general solution.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y = 3px + 6p^2y^2$
2. $y = 2px + p^2y$
3. $p^3 - 4xyp + 8y^2 = 0$
4. $y^2 \log y = xyp + p^2$
5. $x = y + a \log p$
6. $x = y + p^2$

Answers

1. $y^3 = 3cx + 6c^2$
2. $y^2 = 2cy + c^2$
3. $64y = c(c - 4x)^2$
4. $\log y = cx + c^2$
5. $x = c + a \log \frac{p}{p-1}$, $y = c - a \log (p - 1)$
6. $x = -2p - 2 \log (1 - p) + c$, $y = -p^2 - 2p - 2 \log (1 - p) + c$.

3.5 CLAIRAUT'S EQUATION

An equation of the form $$y = px + f(p) \qquad ...(1)$$
is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$, we have $\frac{dp}{dx} = 0$

Integrating, $p = c$

Putting $p = c$ in (1), the required solution is $p = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve $(y - px)(p - 1) = p$.*

Sol. The given equation can be written as

$$y - px = \frac{p}{p-1} \quad \text{or} \quad y = px + \frac{p}{p-1}$$

This is of Clairaut's form. Hence putting c for p, the solution is $y = cx + \frac{c}{c-1}$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Solve for p

Q1. $p^2 + 2py\cot x = y^2$

Soln. $p^2 + 2py\cot x = y^2$

$p^2 + 2py\cot x - y^2 = 0$

$\cancel{p^2 = -2py}$

$$p\ \cancel{p^2} = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{1+\cot^2 x}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\operatorname{cosec}^2 x}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$p = -y\cot x + \operatorname{cosec} x\, y,\ -y\cot x - \operatorname{cosec} x \cdot y$$

$$p = -y\frac{\cos x}{\sin x} + \frac{y}{\sin x},\ -y\frac{\cos x}{\sin x} - \frac{y}{\sin x}$$

$$p = \frac{y(1 \mp \cos x)}{\sin x} \text{ and } -y\left(\frac{1+\cos x}{\sin x}\right)$$

$$p = y \cdot \frac{\cancel{2}\sin^2 x/2}{\cancel{2}\sin\frac{x}{2}\cos x/2} \text{ and } -y \cdot \frac{\cancel{2}\cos^2 x/2}{\cancel{2}\sin x/2 \cos x/2}$$

$p = y\tan x/2$ and $-\cot x/2$

$$\frac{dy}{dx} = y\frac{\sin x/2}{\cos x/2} \text{ and } \frac{dy}{dx} = -y\frac{\cos x/2}{\sin x/2}$$

$\frac{dy}{y} = \frac{\sin x/2}{\cos x/2}\,dx$ and $\frac{dy}{y} = -\frac{\cos x/2}{\sin x/2}\,dx$

Integrating on both sides

$\int \frac{dy}{y} = -2\int \frac{\left(-\frac{1}{2}\sin x/2\right)}{\cos x/2}\,dx$ and $-2\int \frac{\frac{1}{2}\cos x/2}{\sin x/2}\,dx = \int \frac{dy}{y}$

$\log y = -2\log \cos x/2 + \log c_1$ and $\log y = -2\log \sin x/2 + \log c_2$

$\log y = \log \frac{c_1}{\cos^2 x/2}$ and $\log y = \log \frac{c_2}{\sin^2 x/2}$

$\Rightarrow y\cos^2 x/2 = c_1$ and $y\sin^2 x/2 = c_2$

$\Rightarrow$ ~~G~~ General Solution is

$(y\cos^2 x/2 - c_1)(y\sin^2 x/2 - c_2) = 0$

or we can write

$\left(y\left(\frac{1+\cos x}{2}\right) - c_1\right)\left(y\left(\frac{1-\cos x}{2}\right) - c_2\right) = 0$

$(y(1+\cos x) - c)(y(1-\cos x) - c) = 0$

Q2 $p^2 - 2p\cosh x + 1 = 0$

[Using formula
$\cosh^2 x - \sinh^2 x = 1$]

$p = \frac{2\cosh x \pm \sqrt{4\cosh^2 x - 4}}{2}$

$= \frac{2\cosh x \pm 2\sqrt{\cosh^2 x - 1}}{2}$

$= \cosh x \pm \sqrt{\sinh^2 x}$

$= \cosh x \pm \sinh x$

$P = \cosh x + \sinh x$ and $P = \cosh x - \sinh x$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$P = e^{x}$ and $P = e^{-x}$

$\dfrac{dy}{dx} = e^{x}$ and $\dfrac{dy}{dx} = e^{-x}$

$dy = e^{x} dx$ and $dy = e^{-x} dx$

Integrating on both sides

$y = e^{x} + c_1$ and $y = -e^{-x} + c_2$

$\Rightarrow$ general Solution is

$(y - e^{x} - c)(y + e^{-x} - c) = 0$

Solve for y

a $y = 3x + \log p$

The given differential equation is

$y = 3x + \log P$

Differentiate both sides w.r.t x

$\dfrac{dy}{dx} = 3 + \dfrac{1}{P}\dfrac{dP}{dx}$

Put $\dfrac{dy}{dx} = P$

$P - 3 = \dfrac{1}{P}\dfrac{dP}{dx}$

$P(P-3) = \dfrac{dP}{dx}$

$$dx = \frac{dP}{P(P-3)}$$

$$dx = \frac{1}{3}\left[\frac{1}{P-3} - \frac{1}{P}\right] dP$$

Integrating both sides w.r.t x

$$x = \frac{1}{3}\left[\log(P-3) - \log P\right] + \log c$$

$$x = \frac{1}{3}\left[\log\left(\frac{P-3}{P}\right)\right] + \log c$$

$$3x = \log\left(\frac{P-3}{P} \cdot c\right)$$

$$\Rightarrow \frac{P-3}{P} \cdot c = e^{3x}$$

$$\left(\frac{P}{P} - \frac{3}{P}\right) c = e^{3x}$$

$$\left(1 - \frac{3}{P}\right) c = e^{3x}$$

$$c - \frac{3c}{P} = e^{3x}$$

$$c - e^{3x} = \frac{3c}{P}$$

$$\Rightarrow P = \frac{3c}{c - e^{3x}}$$

Putting value of P in given equation

$$y = 3x + \log\frac{3c}{c - e^{3x}}$$

Q $y = p\sin p + \cos p$

Soln Taking derivative w.r.t x

$$\frac{dy}{dx} = p\cos p\frac{dp}{dx} + \frac{dp}{dx}\sin p - \sin p\frac{dp}{dx}$$

$$\Rightarrow p = p\cos p\frac{dp}{dx}$$

$$dx = \cos p\, dp$$

Integrating on both Sides

$$x = \sin p + c$$

$$\sin p = x - c$$

$$\Rightarrow p = \sin^{-1}(x-c)$$

Putting value of p in given equation

$$y = \sin^{-1}(x-c)\cdot(x-c) + \cos[\sin^{-1}(x-c)]$$

Q $p^3 + p = e^y$

Soln. $p(p^2+1) = e^y$

taking log on both Sides

$$\log p(p^2+1) = \log e^y$$

$$\log p + \log(p^2+1) = \log e^y = y\log e$$

$$\Rightarrow y = \log p + \log(p^2+1)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{p}\frac{dp}{dx} + \frac{1}{p^2+1} \, 2p \frac{dp}{dx}$$

Put $\frac{dy}{dx} = p$

$$p = \frac{dp}{dx}\left(\frac{1}{p} + \frac{2p}{p^2+1}\right)$$

$$dx = dp \, \frac{1}{p}\left(\frac{1}{p} + \frac{2p}{p^2+1}\right)$$

$$dx = dp\left(\frac{1}{p^2} + \frac{2}{p^2+1}\right)$$

taking integration on both sides

$$x = -\frac{1}{p} + 2\tan^{-1} p$$

$\Rightarrow$ General Solution is

$$y = \log p + \log(p^2+1)$$

$$x = -\frac{1}{p} + 2\tan^{-1} p$$

Q $16x^2 + 2p^2y - p^3x = 0$

Soln. $2p^2y = p^3x - 16x^2$

$$y = \frac{p^3x - 16x^2}{2p^2}$$

$$y = \frac{px}{2} - \frac{8x^2}{p^2}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\left[p + x\frac{dp}{dx}\right] - 8\left[\frac{2x}{p^2} - \frac{2x^2}{p^3}\frac{dp}{dx}\right]$$

$$p = \frac{p}{2} + \frac{x}{2}\frac{dp}{dx} - \frac{16x}{p^2} + \frac{16x^2}{p^3}\frac{dp}{dx}$$

$$p - \frac{p}{2} + \frac{16x}{p^2} = \left[\frac{x}{2} + \frac{16x^2}{p^3}\right]\frac{dp}{dx}$$

$$\frac{p}{2} + \frac{16x}{p^2} = x\left[\frac{1}{2} + \frac{16x}{p^3}\right]\frac{dp}{dx}$$

$$\frac{(\cancel{p^3 + 32x})}{2p^2} = x\left[\frac{\cancel{p^3 + 32x}}{2p^3}\right]\frac{dp}{dx}$$

$$\frac{1}{\cancel{2}p^2} = \frac{x}{\cancel{2}p^3}\frac{dp}{dx}$$

$$1 = \frac{x}{p}\frac{dp}{dx}$$

$$\frac{dx}{x} = \frac{dp}{p}$$

Integrating on both sides

$$\log x = \log p + \log c$$

$$\log x = \log pc$$

$$\Rightarrow x = pc$$

$$\Rightarrow p = x/c$$

put p in given equation

$$y = \frac{x^2}{2c} - \frac{8x^2}{x^2/c^2}$$

$$y = \frac{x^2}{2c} - 8c^2$$, is required general solution

(c) $y = 1 + xp^3$

Soln. Differentiate w.r.t x both sides

$$\frac{dy}{dx} = 0 + p^3 + x \cdot 3p^2 \frac{dp}{dx}$$

$$p = p^3 + 3xp^2 \frac{dp}{dx}$$

$$p = p^2 \left(p + 3x \frac{dp}{dx}\right)$$

$$1 = p\left(p + 3x \frac{dp}{dx}\right)$$

$$1 - p^2 = 3xp \frac{dp}{dx}$$

$$\frac{1-p^2}{p} = 3x \frac{dp}{dx}$$

$$\frac{dx}{x} = 3 \frac{p}{1-p^2} dp$$

$$\frac{dx}{x} = 3 \frac{p}{(1-p)(1+p)} dp$$

$$\frac{dx}{x} = 3 \frac{1}{2}\left[\frac{1}{1-p} - \frac{1}{1+p}\right] dp$$

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Integrating on both sides

$$\log x = \frac{3}{2}[\log(1-P) - \log(1+P)] + \log c$$

$$\frac{2}{3}\log x = \log\frac{(1-P)}{1+P}c$$

$$\log x^{2/3} = \log\left(\frac{1-P}{1+P}\right)c$$

$$\Rightarrow x^{2/3} = \left(\frac{1-P}{1+P}\right)c$$, its required Solution.

Solve for x

Q $y = 3Px + 6P^2y^2$

$$3Px = y - 6P^2y^2$$

$$x = \frac{y}{3P} - \frac{6P^2y^2}{3P}$$

$$x = \frac{y}{3P} - 2Py^2$$

Differentiate w.r.t y on both sides

$$\frac{dx}{dy} = \frac{1}{3}\left[\frac{1}{P} + y\left(-\frac{1}{P^2}\right)\frac{dP}{dy}\right] - 2\left[2yP + y^2\frac{dP}{dy}\right]$$

$$\frac{dx}{dy} = \frac{1}{3P} - \frac{y}{3P^2}\frac{dP}{dy} - 4yP - 2y^2\frac{dP}{dy}$$

Put $\frac{dx}{dy} = \frac{1}{P}$

$$\frac{1}{p} = \frac{p}{3p} - \frac{dp}{dy}\, y\left[\frac{1}{3p^2} + 2y\right] - 4yp$$

$$\frac{1}{p} - \frac{1}{3p} = -\frac{dp}{dy}\left[\frac{1+6p^2y}{3p^2}\right]y - 4yp$$

$$\frac{1}{p} - \frac{1}{3p} + 4yp = -\frac{dp}{dy}\left[\frac{1+6p^2y}{3p^2}\right]y$$

$$\frac{2}{3}\frac{1}{p} + 4yp = -\frac{dp}{dy}\left[\frac{1+6p^2y}{3p^2}\right]y$$

$$\frac{2\cancel{(1+6yp^2)}}{\cancel{3}p} = -\frac{dp}{dy}\left[\frac{\cancel{1+6p^2y}}{\cancel{3}p^2}\right]y$$

$$2 = -\frac{dp}{dy}\,\frac{y}{p}$$

$$\frac{2\,dy}{y} = -\frac{dp}{p}$$

Integrating on both Sides

$$2\log y = -\log p + \log c$$

$$\log y^2 = \log c/p$$

$$\Rightarrow y^2 = \frac{c}{p}$$

$$\Rightarrow p = c/y^2$$

Put in given equation

$$y = 3\cdot\frac{c}{y^2}\,x + 6\,\frac{c^2}{y^4}\cdot y^2$$

$$y^3 = 3cx + 6c^2$$

Q $x = y + a \log p$

Soln. Differentiate w.r.t y

$$\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1-p}{p} = \frac{a}{p}\frac{dp}{dy}$$

$$dy = a\frac{dp}{1-p}$$

$$y = -a\log(1-p) + \log c$$

$$\frac{y}{a} = \cancel{\log(1-p)\cdot c}\ \log\frac{c}{(1-p)}$$

$$\frac{c}{1-p} = e^{y/a}$$

$$\frac{c}{e^{y/a}} = 1-p$$

$$p = 1 - \frac{c}{e^{y/a}}$$

is required solution.